Derivation of the Thrust Equation from Conservation of Energy

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The thrust equation for rockets and air-breathing jets, conventionally derived by use of the momentum theorem of fluid mechanics, is derived here from the first law of thermodynamics. The principle of conservation of energy, applied alternately to a system and to a control volume, yields the thrust equation quite directly. The particular control volume specified is compared to two others which are commonly used in derivations based on the momentum theorem. Intermediate relations developed in the analysis are shown to be useful for study of the over-all energy balance of propulsive systems and for interpretation of several definitions of propulsive efficiency used for air-breathing systems.

Nomenclature

A_e = exit area of thrust producer
f = fuel-air mixture ratio by mass = $\dot{m}_f/\dot{m}_a$
F = thrust
g = acceleration resulting from gravity
h = enthalpy per unit mass
$\dot{m}$ = mass flow rate
p = pressure
$\dot{Q}$ = heat added per unit time
u = absolute vehicle velocity
u_e = velocity of exhaust fluid relative to thrust producer and in the direction of vehicle motion
v = absolute velocity in the direction normal to vehicle motion
V = velocity, absolute or relative as specified
$\dot{W}$ = work accomplished per unit time
z = elevation above datum level
η_p = propulsive efficiency

Subscripts

a = air entering system or control volume or freestream condition
e = exit of thrust producer
f = fuel at fuel inlet conditions
n = direction normal to vehicle motion
u = direction of vehicle motion

Introduction

The thrust of rockets and air-breathing jets is classically derived by application of the momentum theorem of fluid mechanics.[1–10] Use of Newton's laws of motion for this purpose is an entirely reasonable procedure since it is customary to think of jets as operating on the reaction principle. Then too, the laws of motion should have a certain priority for they were formulated by Newton 150 years before enunciation of the first law of thermodynamics or the law of conservation of energy from which the thrust equation may be just as simply and rigorously derived, as will be demonstrated in this paper. The author is unaware of any previous formal derivation of the thrust equation which does not in some manner use the momentum equation of fluid mechanics. On the other hand, an understanding of the general principles involved in the formulation of the thrust equation from conservation of energy alone is certainly implied in a number of energy balance diagrams and definitions of efficiencies which appear in the literature.[6,9]

The advantage of the energy equation approach is that it not only yields the thrust equation in a direct manner but it also gives at once an understanding of the energy balance and hence, some insight into the significance of the somewhat arbitrary definitions of efficiencies that are applied to thrust-producing devices. The derivation also serves to illustrate the difference between a thermodynamic system and a control volume analysis as used in fluid mechanics. Thus, at least as a pedagogical device, the approach seems to warrant formal development.

Common to the derivations of the thrust equation which rest on the momentum theorem is at least an implicit assumption that the flow external to the thrust-producing device is frictionless. With this assumption, it is not surprising that the thrust equation may be derived from energy considerations alone. It is well known that in the absence of fluid shear, capillarity, electricity, and magnetism, the steady-flow energy equation of compressible fluid flow and the corresponding momentum equation (Bernoulli's equation) are the same.[11] Thus, if the manifestations of internal viscosity can be eliminated from an energy balance for the thrust-producing system without destroying the balance, the result should yield the thrust equation. This is accomplished by applying the energy equation in two ways: first, to a system which consists of the thrust-producing device and certain fluid elements and second, to a control volume which moves with the thrust producer. Combination of the two equations yields the thrust equation.

The Energy Equation for the System

Figure 1 depicts a generalized thrust-producing device. Although the device looks like a pylon-mounted turbojet, the results obtained will be applicable to any air-breathing propulsion system having a single exhaust stream, and with simplification, also applicable to rockets. In Fig. 1, a boundary surface is specified which passes through the nozzle exhaust plane at e, extends upstream across the outer surfaces of the thrust-producing device cutting the structural support in a plane parallel to the direction of motion of the vehicle, and then extends far upstream along the streamlines bordering the entering flow where in the unperturbed flow a plane normal to the streamlines completes the enclosure at a. A thermodynamic system is now defined consisting principally of the material within the boundary surface. Included in the system is the infinitesimal amount of air dm_a at plane a which in an infinitesimal time increment will pass inside the boundary described previously; excluded from the system is the infinitesimal amount of exhaust products which in the same time interval will pass outward across the boundary. It is assumed that conditions within the specified boundary do not change with time. The first law of thermodynamics applied to the system gives

$$dQ - dW = -\Sigma(h + V^2/2g + z)\delta m \quad (1)$$

where dQ and dW denote, respectively, the heat passing inward and the work passing outward across the boundary when

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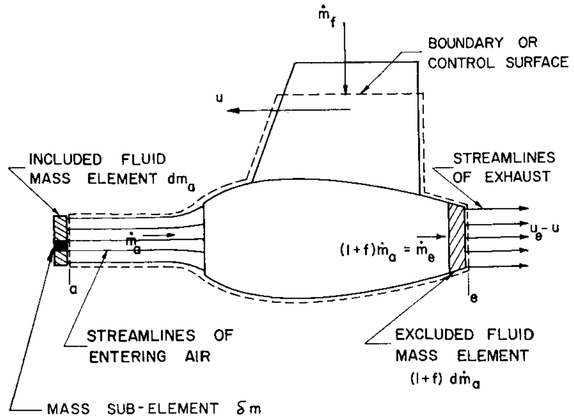

Fig. 1 Generalized thrust-producing device.

the mass element dm_a enters, and the summation represents the net change in enthalpy, kinetic energy, and potential energy for the mass subelements δm which cross the boundary while the mass element is entering.[11] If a mass subelement enters, δm is positive, while for a departing subelement, δm is negative. The following assumptions are common to the usual derivation of the thrust equation by any method: 1) the changes of potential energy within the system are negligible and 2) the fluids are uniform in composition, state, and velocity at the entrance and at the exit.

Equation (1) may now be written on a per unit time basis as

$$\dot{Q} - \dot{W} = \dot{m}_a[(1 + f)(h_e + V_e^2/2g) - (h_a + V_a^2/2g) - f(h_f + V_f^2/2g)] \quad (2)$$

where f is the fuel-air mixture ratio. Next, to avoid undue complication, two restrictive assumptions are made (these restrictions will be partially relaxed later): The exhaust velocity is axially directed, and the fuel enters with negligible velocity with respect to the vehicle.

In Eq. (2), $\dot{W}$ is the work associated with displacement of the system's boundary.[11] This work includes that accomplished by displacement of the reaction forces at the support and that of displacement of viscous and pressure forces on the remaining portion of the boundary. It is customary to neglect viscous forces at the external boundaries regardless of how the boundaries are specified.[2,3] Those viscous forces which act on the external surfaces of the thrust producer itself are charged to aerodynamic drag which the thrust equation does not take into account. Those associated with fluid interfaces are negligible in view of the small velocity gradients involved. These considerations are now included in the present analysis along with the additional assumption that a uniform pressure p_a acts over the system's boundaries except at the exit and support, as shown in Fig. 2. A uniform pressure p_a acting along the streamlines of the flow entering the engine can exist only in the special case where the streamlines are straight and parallel. The assumption, nevertheless, yields a pressure force system equivalent to those more commonly selected for systems or control volumes; for example, that of Hill and Peterson (Fig. 3a) or Durham (Fig. 3b).[2,3] From the principle of hydrostatic equilibrium, the resultant of each of the three external pressure distributions in the direction of motion is $(p_e - p_a)A_e$ where A_e is the exit area of the thrust producer and p_e is the pressure in the jet stream at the exit plane. Use of the present boundary obviates the need to take into account mass flow through the sides of the system as must be done in the case of the control volume shown in Fig. 3a. Nor is it necessary to separately add the exterior pressure distribution as must be done for Fig. 3b.

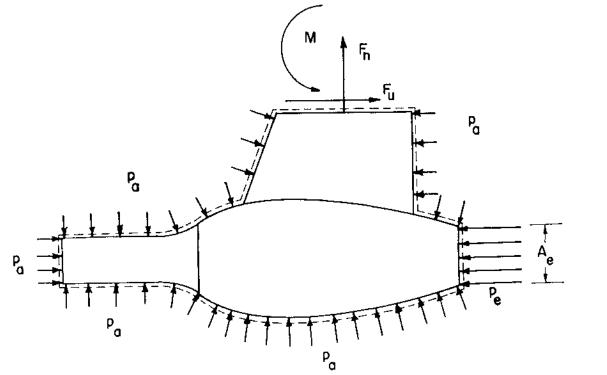

Fig. 2 Force system acting at the boundaries of system of Fig. 1.

Equation (2) may now be written as

$$\dot{Q} - F_u u - (p_a - p_e)A_e u = \dot{m}_a\{(1 + f)[h_e + (u_e - u)^2/2] - h_a - f(h_f + u^2/2)\} \quad (3)$$

The second term of Eq. (2) represents the work accomplished per unit time by the system as a result of the reaction force at the support. Only the thrust force F_u in the direction of motion does work. It is important to recognize that absolute velocities must be used in applying Eq. (1). Thus, the velocity applicable to the exhaust fluid is the velocity u_e of the fluid with respect to the thrust producer less the vehicle velocity u. The velocity of the entering air is zero, and the entering fuel has the vehicle velocity. The symbol for velocity has been changed from V to u to emphasize that under the current assumptions only axially directed velocities exist at the inlets and outlet.

Energy Equation for the Control Volume

The boundary surface previously described is now adopted for a control volume of fixed shape which is attached to and

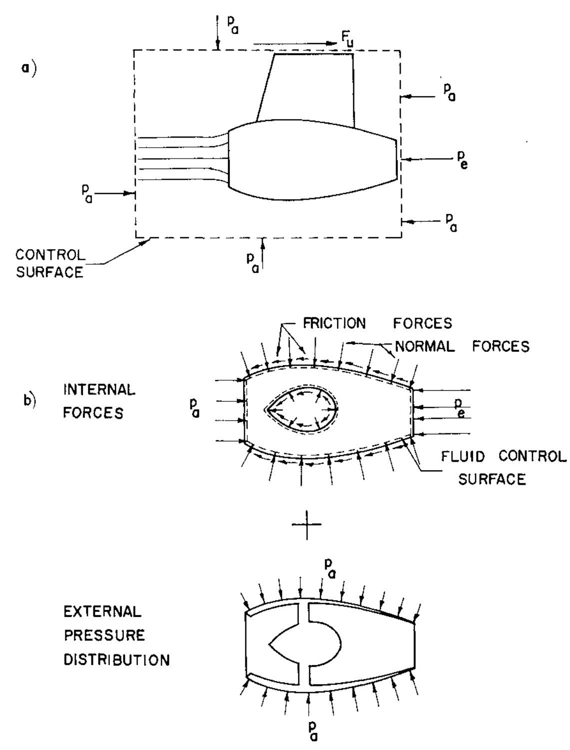

Fig. 3 Two alternate control volumes and assumed external pressure distributions.

moves with the thrust producer. Derivation of the steady-flow energy equation for a control volume may be found in a number of excellent references, e.g., Shames.[7] The essential difference between the energy equation for the system and for the control volume is that for the latter, velocities and time derivatives are measured with respect to the moving control volume. Indeed, with this qualification, the energy equation for the control volume is the same as Eq. (2) for the system. However, now Eq. (2), upon substitution of relative velocities and evaluation of the time derivative of the work accomplished, takes the form

$$\dot{Q} = \dot{m}_a[(1 + f)(h_e + u_e^2/2) - (h_a + u^2/2) - fh_f] \quad (4)$$

Thrust Equation and Propulsive Efficiency

It is only necessary to recognize that the heat-transferred-per-unit mass of air entering the system and the heat-transferred-per-unit mass of air entering the control volume are the same to permit elimination of the heat term between Eqs. (3) and (4). This combination gives

$$\dot{m}_a[(1 + f)(h_e + u_e^2/2) - (h_a + u^2/2) - fh_f] - F_u u - (p_a - p_e)A_e u = \dot{m}_a\{(1 + f)[h_e + (u_e - u)^2/2] - h_a - f(h_f + u^2/2)\} \quad (5)$$

which after simplification yields the thrust equation

$$F_u = \dot{m}_a[(1 + f)u_e - u] + (p_e - p_a)A_e \quad (6)$$

For a rocket, $\dot{m}_a$ is zero and the thrust equation reduces to

$$F_u = \dot{m}_f u_e + (p_e - p_a)A_e \quad (7)$$

Now, Eqs. (6) and (7) are based on the assumption of axially directed exhaust velocity while in the momentum derivation this assumption is not necessary provided u_e is taken as the axial component of the exhaust velocity. However, reexamination of the energy equation procedure shows that if a component of the exhaust velocity v_e exists normal to the direction of motion, a term $\dot{m}_a(1 + f)v_e^2/2$ must be added to the right-hand sides of both Eqs. (3) and (4). These terms cancel each other when the two equations are combined. The same reasoning may be applied to show that it is necessary to neglect only the axial component of the entering fuel velocity.

Equation (3) represents a complete energy balance for the system which aids in appreciation of the significance of the various definitions of efficiencies. For example, the propulsive efficiency η_p for an air-breathing jet is frequently defined as the ratio of thrust power, $F_u u$, to the so-called rate of production of propellant kinetic energy.[3] This ratio is written

$$\eta_p = F_u u/\dot{m}_a[(1 + f)u_e^2/2 - u^2/2] \quad (8)$$

An interpretation of the propulsive efficiency may be obtained by rewriting Eq. (4):

$$(1 + f)u_e^2/2 - u^2/2 = \dot{Q}/\dot{m}_a + h_a + fh_f - (1 + f)h_e \quad (9)$$

This shows that the denominator of Eq. (8) may be expressed in terms of the thermal energy rates involved. The latter quantities are easily identified in the over-all power balance, Eq. (3). In the special case where $p_e = p_a$ and $f \ll 1$, with the aid of Eqs. (3) and (9), the stated definition of propulsive efficiency may also be rewritten as

$$\eta_p = F_u u/[F_u u + \dot{m}_a(u_e - u)^2/2] \quad (10)$$

This means that the basis of the stated definition of propulsive efficiency is the thrust power plus a quantity which is essentially the residual kinetic energy rate of the exhaust gases, and the propulsive efficiency is sometimes defined directly as such for air-breathing jets.[4,10] This word definition is the common one applied to rockets. It is significant that for air-breathing jets the two definitions, Eqs. (8) and (10), are consistent only when $p_e = p_a$ and f may be ignored.

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Aircraft Handling, Stability, and Control; Navigation, Control, and Guidance Theory

Theme

The theory and application of a new performance index, the Model PI, that brings engineering design specifications into the analytical design process is presented. A parameter optimization procedure is used to obtain a fixed configuration design that meets practical engineering specifications. The technique is demonstrated by designing a lateral-directional stability augmentation system for the X-15 aircraft.

Content

The linearized, single-input, single-output transfer characteristics of an aircraft with a stability augmentation system (SAS) can be represented completely as the response of a linear autonomous system

$$\mathbf{x}'(t)\mathbf{a} = 0 \tag{1}$$

to a set of pseudo initial conditions $\mathbf{x}_0$, where $\mathbf{x}(t)$ is the extended phase variable state vector, $\mathbf{a}$ is a vector whose elements are the coefficients of the system's characteristic equation, and $\mathbf{x}_0$ is a vector of pseudo-initial conditions that would produce the same transient response as the original transfer function for unit step input. The Model PI is defined as

$$PI = \int_0^\infty ||\mathbf{x}(t)||^2 \mathbf{Q} dt \tag{2}$$

where

$$Q = \alpha\alpha'/||\alpha||^2 \tag{3}$$

Minimizing Eq. (2) with espect to the design parameters tends to force the system's response $\mathbf{x}(t)$ to approximate a response $\mathbf{x}_m(t)$ given by

$$x'_m(t)\alpha = 0 \tag{4}$$

Therefore, a model of the desired response Eq. (4) can be included in a quadratical functional of the Eq. (2) simply by selecting an appropriate weighting matrix of the Eq. (3). This not only gives a physical interpretation to the quadratic weighting matrix but also reduces the computational task of the parameter optimizations procedure over the familiar model-referenced integral squared error approach.

The Model PI theory is developed on the basis of a novel geometrical criterion for approximating one dynamical system by another. It applies to models of equal or lower order than the system and to multivariable systems. A parameter optimization design procedure is established that starts with practical engineering specifications and uses the Model PI as a synthesis tool to obtain a satisfactory design. The example presented shows that it can be used effectively in designing practical flight control systems.